Exam Sets & Numbers (part 2: integers and modular arithmetic),
November 8th, 2023, 8:30am–10:30am,
Aletta Jacobshal 01 S19-Y14.

Write your name on every sheet of paper that you intend to hand in.
Please provide **complete** *arguments for each of your answers.*
This exam consists of 3 *questions.*
You can score up to 6 *points for each question, and you obtain* 2 *points for free.*
In this way you will score in total between 2 *and* 20 *points.*

(1) For $k \in \mathbb{Z}_{\geq 0}$ consider the numbers $a_k := (42 \cdot 10^k - 6)/18$.
 (a) [2 points]. Explain why $a_k \in \mathbb{Z}$ for all $k \in \mathbb{Z}_{\geq 0}$.
 (b) [2 points]. Prove that $11 \nmid a_k$ holds for all $k \in \mathbb{Z}_{\geq 0}$.
 (c) [2 points]. Show that $13 | (a_{8112023} + 4)$.

(2) (a) [3 points]. Take $n \in \mathbb{Z}$ such that $2 \nmid n$, and let $a \in \mathbb{Z}$ satisfy $\gcd(a, n) = 1 = \gcd(a + 2, n)$. Prove that in $\mathbb{Z}/n\mathbb{Z}$ the element $\overline{(a)}^{-1} - \overline{(a+2)}^{-1}$ is a unit.
 (b) [3 points]. Determine the number of positive integers n between 0 and 420 such that n is a unit modulo 70.

(3) (a) [3 points]. Compute the last 3 decimal digits of 3^{402}.
 (Hint: it helps to first determine $\phi(1000)$, with ϕ denoting the Euler phi function.)
 (b) [3 points]. Suppose $n, m \in \mathbb{Z}$ satisfy $\gcd(n, m) = 1$. Prove that $nm | \left(n^{\phi(m)} + m^{\phi(n)} - 1\right)$.